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Contour-improved versus fixed-order perturbation theory in hadronic τ decays

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Abstract: The hadronic decay rate of the τ lepton serves as one of the most precise determinations of the QCD coupling α_s. The dominant theoretical source of uncertainty at present resides in the seeming disparity of two approaches to improving the perturbative expansion with the help of the renormalisation group, namely fixed-order and contour-improved perturbation theory. In this work it is demonstrated that in fact both approaches yield identical results. However, the fixed-order series is found to oscillate around the contour-improved result with an oscillation frequency of approximately six perturbative orders, approaching it until about the 30th order, after which the expansion reveals its asymptotic nature.

Keywords: Sum Rules, QCD.

Contents

1. Introduction

Already more than a decade ago it was realised that the hadronic decay of the τ lepton could serve as an ideal system to study low-energy QCD under rather clean conditions [1–4]. In the following years, detailed investigations of the τ hadronic width as well as invariant mass distributions have served to determine the QCD coupling α_s to a precision competitive with the current world average [5–9]. The experimental separation of the Cabibbo-allowed decays and Cabibbo-suppressed modes into strange particles [10–12] opened a means to also determine the quark-mixing matrix element $|V_{us}|$ [13–15] as well as the mass of the strange quark [16–23], additional fundamental parameters within the Standard Model.

The starting point for a QCD analysis of the τ hadronic width R_τ is the finite energy sum rule (FESR) [1, 24]

$$\frac{\Gamma(\tau \to \text{hadrons}\,\nu_\tau)}{\Gamma(\tau \to \mu\bar{\nu}_\mu\nu_\tau)} \equiv R_\tau^\Pi(s_0) = \int_0^{s_0} w_\tau(s)\rho_\tau(s)ds = \frac{-1}{2\pi i}\oint\limits_{|s|=s_0} w_\tau(s)\Pi_\tau(s)\,ds\,. \tag{1.1}$$

The FESR can easily be derived from Cauchy's theorem and the fact that the correlation function $\Pi_\tau(s)$ is analytic in the whole complex s-plane, except for the positive real axis, where it may have poles and cuts. The correlator $\Pi_\tau(s)$ of hadronic QCD currents receives contributions from vector and axialvector correlation functions for both (ud) as well as (us) flavour content, dressed with the appropriate quark mixing matrix factors:

$$\Pi_\tau(s) = |V_{ud}|^2\left[\Pi_{ud}^{\rm V}(s) + \Pi_{ud}^{\rm A}(s)\right] + |V_{us}|^2\left[\Pi_{us}^{\rm V}(s) + \Pi_{us}^{\rm A}(s)\right]. \tag{1.2}$$

Furthermore, $\rho_\tau(s) \equiv \mathrm{Im}\,\Pi_\tau(s+i0)/\pi$ is the so-called spectral function which contains all the physical information.

In the case of hadronic τ decays, s is the invariant mass of the final hadron system, $w_\tau(s)$ is the relevant kinematic weight function,

$$w_\tau(s) = \frac{12\pi^2}{s_0}\left(1-\frac{s}{s_0}\right)^2\left(1+2\frac{s}{s_0}\right), \tag{1.3}$$

and $s_0 = M_\tau^2$, though for the moment the somewhat more general expression will be kept. The fortuitous fact about R_τ is that the weight function introduces a double zero at $s = s_0$, such that there is no need to evaluate $\Pi_\tau(s)$ close to the real axis where perturbation theory becomes problematic. In writing eq. (1.1), a small contribution from longitudinal correlators has been omitted. Being suppressed by the light quark masses, it is of no relevance for the purpose of this work which will concentrate on the massless correlators, but it will be reconsidered in a subsequent publication.

Since the hadronic τ decay rate is a physical, measurable quantity, the corresponding perturbative QCD expression can be improved with the help of the renormalisation group equation (RGE). In the course of calculating $R_\tau(s_0)$, one also has to perform the contour integration over the circle with radius s_0 in the complex s-plane. Then the question arises in which order both operations should be performed. First calculating the contour integral and then performing the renormalisation group improvement goes under the name of *fixed-order* perturbation theory (FOPT) whereas the second approach of first resumming the expansion and afterwards integrating over the contour has been suggested in [25, 26] and is termed *contour-improved* perturbation theory (CIPT). Both approaches will be investigated in much detail below.

Numerically, it is found that the CI perturbative expansion displays a better behaviour than the corresponding FO series [26]. In particular, employing the known perturbative results up to order α_s^3 as well as estimates for the contributions at order α_s^4 and α_s^5, it is found that the difference between the two approaches is much larger then would be naively expected based on the last included terms in the expansion. In the past, this apparent ambiguity has cast doubts as to which way of resumming the perturbative series is preferable and it represents the dominant source of theoretical uncertainty in the determination of α_s from the hadronic τ decay rate up to this day.

Below, it is demonstrated that in fact both approaches to the renormalisation group improvement of the perturbative series yield identical results. However, the fixed-order series is found to oscillate around the contour-improved result with an oscillation frequency of approximately six perturbative orders, approaching it until about the 30th order, after which the expansion reveals its asymptotic nature. To this end, in the next section, the perturbative expansion of the vector correlation function will be reviewed, and in section 3 the two approaches of improving the perturbative series in the τ-decay finite energy sum rule will be discussed. In section 4, a simplified example will be analysed, in which only the first coefficient of the β-function β_1 is non-vanishing. Whereas this case can be treated fully analytically, it shares all the essential features of the complete QCD expression. The more realistic case which includes all currently known terms in the β-function is then treated in section 5. Finally, in the conclusions, my main findings of this work will be summarised.

2. Perturbative correlator

By far the most important contribution to the hadronic τ decay rate is due to the purely perturbative vector (axialvector) correlation function in the limit of vanishing quark masses [1]. In this limit vector and axialvector correlators yield identical contributions. Therefore, in what follows, only the vector contribution will be investigated. A discussion of quark mass corrections shall be presented in a forthcoming publication, but for the remainder of this work, the quark masses are assumed to be zero.

In the massless case, the perturbative expansion of the vector correlation function $\Pi(s)$ exhibits the general structure:[1]

$$\Pi(s) = -\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{nk}\, L^k \qquad \text{where} \qquad L \equiv \ln\frac{-s}{\mu^2}\,, \tag{2.1}$$

and $a_\mu \equiv \alpha_s(\mu^2)/\pi$ with μ being the renormalisation scale. (The global minus sign has been introduced for later convenience.) $\Pi(s)$ itself is not a physical quantity. However, this is the case for the spectral function $\rho(s)$ as well as for the Adler function $D(s)$:

$$D(s) \equiv -s\frac{d}{ds}\,\Pi(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} L^{k-1}\,. \tag{2.2}$$

It should be clear that the coefficients c_{n0} do not contribute to both the spectral function (because they are real) and to the Adler function (because of the factor k). Thus the coefficients c_{n0}, which actually are found to depend on the renormalisation prescription, can be considered as "unphysical" in that they do not appear in measurable quantities.

As physical quantities, both $D(s)$ as well as $\rho(s)$ have to satisfy a homogeneous RGE:

$$-\mu\frac{d}{d\mu}\begin{Bmatrix} D(s) \\ \rho(s) \end{Bmatrix} = \left[2\,\frac{\partial}{\partial L} + \beta(a)\frac{\partial}{\partial a}\right]\begin{Bmatrix} D(s) \\ \rho(s) \end{Bmatrix} = 0\,, \tag{2.3}$$

where $\beta(a)$ is the QCD β-function, defined as:

$$-\mu\frac{da}{d\mu} \equiv \beta(a) = \beta_1\, a^2 + \beta_2\, a^3 + \beta_3\, a^4 + \beta_4\, a^5 + \cdots\,. \tag{2.4}$$

Numerically, for $N_c = 3$ and in the $\overline{\rm MS}$-scheme [27] the first four coefficients are given by [28, 29]:

$$\begin{aligned}
\beta_1 &= \frac{11}{2} - \frac{1}{3}\,N_f\,, \qquad \beta_2 = \frac{51}{4} - \frac{19}{12}\,N_f\,, \qquad \beta_3 = \frac{2857}{64} - \frac{5033}{576}\,N_f + \frac{325}{1728}\,N_f^2\,, \\
\beta_4 &= \frac{149753}{768} + \frac{891}{32}\,\zeta_3 - \Big(\frac{1078361}{20736} + \frac{1627}{864}\,\zeta_3\Big)N_f + \Big(\frac{50065}{20736} + \frac{809}{1296}\,\zeta_3\Big)N_f^2 + \\
&\quad + \frac{1093}{93312}\,N_f^3\,.
\end{aligned} \tag{2.5}$$

[1] Since in the rest of this work, we will only be concerned with the vector correlator, the superscript V will be dropped, and also the flavour content does not matter in the limit of vanishing quark masses.

The RGE puts constraints on the coefficients c_{nk}. Considering the coefficients c_{n1} to be independent, all other coefficients c_{nk} with $k = 2, \ldots, n+1$ can be expressed in terms of the c_{n1} and β-function coefficients. Up to order α_s^4, the RG constraints lead to:

$$
\begin{aligned}
c_{22} &= -\frac{\beta_1}{4}\, c_{11}\,, & c_{33} &= \frac{\beta_1^2}{12}\, c_{11}\,, \\
c_{32} &= -\frac{1}{4}(\beta_2\, c_{11} + 2\beta_1 c_{21})\,, & c_{44} &= -\frac{\beta_1^3}{32}\, c_{11}\,, \\
c_{43} &= \frac{\beta_1}{24}\,(5\beta_2\, c_{11} + 6\beta_1\, c_{21})\,, & c_{42} &= -\frac{1}{4}\,(\beta_3\, c_{11} + 2\beta_2\, c_{21} + 3\beta_1\, c_{31})\,.
\end{aligned}
\tag{2.6}
$$

Furthermore, the coefficients $c_{n,n+1} = 0$ for $n \geq 1$. The independent coefficients c_{n1} are known analytically up to order α_s^3 [30, 31] and at $N_c = 3$ take the following values:

$$
\begin{aligned}
c_{01} &= c_{11} = 1\,, \qquad c_{21} = \frac{365}{24} - 11\zeta_3 - \Big(\frac{11}{12} - \frac{2}{3}\zeta_3\Big)N_f\,, \\
c_{31} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 - \Big(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\Big)N_f + \Big(\frac{151}{162} - \frac{19}{27}\zeta_3\Big)N_f^2\,.
\end{aligned}
\tag{2.7}
$$

For the next five- and six-loop coefficients c_{41} and c_{51}, estimates employing principles of "minimal sensitivity" (PMS) or "fastest apparent convergence" (FAC), together with recently computed terms of order $\alpha_s^4\, N_f^2$, exist which read [32]:

$$
c_{41} = 27 \pm 16\,, \qquad c_{51} = 145 \pm 100\,. \tag{2.8}
$$

For illustrative purposes, also the central values of these estimates will be taken into account in the analysis below.

Since the Adler function $D(s)$ satisfies a homogeneous RGE, the logarithms in eq. (2.2) can be resummed with the choice $\mu^2 = -s \equiv Q^2$, leading to the simple expression:

$$
D(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} c_{n1}\, a_Q^n\,, \tag{2.9}
$$

where $a_Q \equiv \alpha_s(Q^2)/\pi$. As is again apparent from this equation, the only physically relevant coefficients are the c_{n1}.

3. Hadronic τ decay rate

In principle, now one could proceed by inserting the general expression (2.1) for $\Pi(s)$ into the contour integral of eq. (1.1). However, it is advantageous to rewrite the FESR for the τ decay rate in terms of the Adler function $D(s)$ by partial integration:[2]

$$
R_\tau^D(s_0) = -6\pi i \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,D(s_0 x)\,, \tag{3.1}
$$

[2] In writing eq. (3.1), $|V_{ud}|^2 + |V_{us}|^2 = 1$ has been assumed, which numerically is satisfied rather well.

with the new dimensionless integration variable being $x \equiv s/s_0$. For FOPT, $R_\tau^\Pi(s_0)$ and $R_\tau^D(s_0)$ can be shown to be identical as they should, but applying CIPT, the perturbative expansion for $R_\tau^\Pi(s_0)$ is less well behaved than the one for $R_\tau^D(s_0)$. Therefore, only the comparison of $R_\tau^D(s_0)$ in FOPT and CIPT will be investigated in this work.

Inserting the expression (2.2) for $D(s)$ into the contour integral of eq. (3.1), one finds:

$$\begin{aligned} R_\tau^D(s_0) &= \frac{N_c}{2\pi i}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\ln^{k-1}\!\left(\frac{-s_0 x}{\mu^2}\right) \\ &= \frac{N_c}{2\pi i}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \sum_{l=0}^{k-1}\binom{k-1}{l}\ln^{k-l-1}\frac{s_0}{\mu^2}\oint_{|x|=1}\frac{dx}{x}\,(1-x)^3\,(1+x)\ln^l(-x)\,, \\ &= N_c\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \sum_{l=0}^{k-1}\binom{k-1}{l} J_l\,\ln^{k-l-1}\frac{s_0}{\mu^2}\,. \end{aligned} \tag{3.2}$$

In the last line, the contour integrals J_l are defined by

$$J_l \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-x)^3\,(1+x)\ln^l(-x) = \frac{1}{2\pi}\left[I_{l,0} + 2\,I_{l,1} - 2\,I_{l,3} - I_{l,4}\right], \tag{3.3}$$

with the required integrals $I_{l,m}$ being given by

$$I_{l,m} \equiv i^l\int_{-\pi}^{\pi}\alpha^l e^{im\alpha}\,d\alpha = (-1)^{l+m}\frac{2\,l!}{m^{l+2}}\sum_{k=1}^{[(l+1)/2]}(-1)^k\frac{m^{2k}\pi^{2k-1}}{(2k-1)!}\,, \tag{3.4}$$

where $[n]$ is denoting the integer part of n and $m \geq 1$. For the case $m = 0$, one obtains $I_{l,0} = i^l[1+(-1)^l]\,\pi^{l+1}/(l+1)$. Explicitly, the first few of the integrals J_l, which are needed up to order α_s^3, read:

$$J_0 = 1\,, \qquad J_1 = -\frac{19}{12}\,, \qquad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2\,, \qquad J_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2\,, \tag{3.5}$$

in agreement with ref. [26].

As discussed above, the Adler function $D(s)$ is a physical quantity, and thus satisfies a homogeneous RGE. Therefore, the logarithms in eq. (3.2) can be resummed. Let us first concentrate on FO perturbation theory, which amounts to taking the choice $\mu^2 = s_0$ in the last line of (3.2), which then leads to

$$R_\tau^{D,\mathrm{FO}}(s_0) = N_c\sum_{n=0}^{\infty} a_{s_0}^n \sum_{k=1}^{n+1} k\, c_{nk}\, J_{k-1}\,, \tag{3.6}$$

with $a_{s_0} \equiv a(s_0)$. Inserting the relations (2.6) for the c_{nk} together with the corresponding contour integrals J_{k-1}, up to order $a_{s_0}^4$, this leads to:

$$\begin{aligned} R_\tau^{D,\mathrm{FO}}(s_0) = N_c\Big\{ & c_{01} + c_{11}\,a_{s_0} + \left(c_{21} + \frac{19}{24}\beta_1 c_{11}\right) a_{s_0}^2 + \\ & + \left(c_{31} + \frac{19}{12}\beta_1 c_{21} + \left(\frac{19}{24}\beta_2 + \left(\frac{265}{288} - \frac{\pi^2}{12}\right)\beta_1^2\right) c_{11}\right) a_{s_0}^3 + \end{aligned}$$

$$+\left(c_{41}+\frac{19}{8}\beta_1 c_{31}+\left(\frac{19}{12}\beta_2+\left(\frac{265}{96}-\frac{\pi^2}{4}\right)\beta_1^2\right)c_{21}+\right. \tag{3.7}$$

$$\left.\left.+\left(\frac{19}{24}\beta_3+\left(\frac{1325}{576}-\frac{5\pi^2}{24}\right)\beta_2\beta_1+\left(\frac{3355}{2304}-\frac{19}{96}\pi^2\right)\beta_1^3\right)c_{11}\right)a_{s_0}^4\right\}$$

$$= N_c\left\{1+a_{s_0}+\left(\frac{769}{48}-9\zeta_3\right)a_{s_0}^2+\left(\frac{363247}{1152}-\frac{27}{16}\pi^2-\frac{2071}{8}\zeta_3+\frac{75}{2}\zeta_5\right)a_{s_0}^3+\right.$$

$$+\left(\frac{19907171}{6144}+c_{41}-\frac{22683}{256}\pi^2-\left(\frac{345405}{128}-\frac{729}{16}\pi^2\right)\zeta_3+\right.$$

$$\left.\left.+\frac{12825}{32}\zeta_5\right)a_{s_0}^4\right\}. \tag{3.8}$$

For the second step, the coefficients of the β-function (2.5) as well as the c_{n1}-coefficients of eq. (2.7) have been employed. An important remark is in order here. As should be clear from eq. (3.7), each order n in FOPT depends on all coefficients c_{i1} with $i \leq n$. This will play a role in what follows. Before analysing the expressions numerically, the second approach of resumming the logarithms, namely CI perturbation theory, will be introduced.

For CIPT, the logarithms in eq. (3.2) should be resummed before performing the contour integral. This can be achieved with the choice $\mu^2 = -s_0 x$ in the first line of (3.2), which yields

$$R_\tau^{D,\mathrm{CI}}(s_0) = N_c \sum_{n=0}^{\infty} c_{n1}\, J_n^a(s_0)\,. \tag{3.9}$$

Here, the equation has been rewritten in terms of the contour integrals $J_n^a(s_0)$, defined as:

$$J_n^a(s_0) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-x)^3\,(1+x)\,a^n(-s_0 x)\,. \tag{3.10}$$

In contrast to the FO case, for CIPT each order n just depends on the corresponding coefficient c_{n1}. Thus, all contributions proportional to the coefficient c_{n1} which in FOPT appear at all perturbative orders equal or greater to n are resummed into a single term. This is related to the fact that CIPT resums the running of the QCD coupling along the integration contour in the complex s-plane. To perform the contour integration, we have to analytically continue the strong coupling α_s to the complex s-plane, but this is straightforward since the dependence of α_s on s is only logarithmic and it has the same cut structure as $\Pi(s)$.

4. A simple example

Before embarking on the general integrals $J_n^a(s_0)$, as a first step, let us investigate a simplified example which can be treated fully analytically, namely the case where only the first coefficient of the β-function, β_1, is non-zero, and all higher order β-coefficients vanish identically. Then the running coupling in eq. (3.10) can be expressed as

$$a^n(-s_0 x) = \frac{a_{s_0}^n}{\left[1+\frac{\beta_1}{2}\,a_{s_0}\ln(-x)\right]^n}\,, \tag{4.1}$$

and the integrals $J_n^a(s_0)$ can be calculated analytically with the result:

$$J_n^a(s_0) = a_{s_0}^n \left[J_{n,0}(\kappa) + 2J_{n,1}(\kappa) - 2J_{n,3}(\kappa) - J_{n,4}(\kappa) \right]. \tag{4.2}$$

Here, the constant $\kappa \equiv \beta_1 a_{s_0}/2$, and the integrals $J_{n,m}(\kappa)$ are found to be

$$J_{n,m}(\kappa) = \frac{\mathrm{e}^{-\frac{m}{\kappa}} \left(\frac{m}{\kappa}\right)^{n-1}}{2\pi i \kappa \Gamma(n)} \left[\mathrm{Ei}\Big(\frac{m}{\kappa} z\Big) - \mathrm{e}^{\frac{m}{\kappa} z} \sum_{l=1}^{n-1} \Gamma(l) \Big(\frac{m}{\kappa} z\Big)^{-l} \right]_{z=1-i\pi\kappa}^{z=1+i\pi\kappa}, \tag{4.3}$$

with $\mathrm{Ei}(z) = \int \mathrm{e}^z/z\, dz$ being the exponential-integral function and $m \geq 1$. Again, the case $m = 0$ which is also required has to be treated separately and yields:

$$J_{1,0}(\kappa) = \frac{1}{2\pi i \kappa} \ln \frac{(1 + i\pi\kappa)}{(1 - i\pi\kappa)}, \qquad n = 1, \tag{4.4}$$

$$J_{n,0}(\kappa) = \frac{1}{2\pi i \kappa (n-1)} \left[(1 - i\pi\kappa)^{1-n} - (1 + i\pi\kappa)^{1-n} \right], \qquad n \geq 2. \tag{4.5}$$

The analytic expressions for the integrals $J_n^a(s_0)$ can readily be compared with their perturbative expansion which is obtained by expanding eq. (4.1) in powers of $\kappa = \beta_1 a_{s_0}/2$, before performing the contour integration:

$$J_n^a(s_0) = \frac{a_{s_0}^n}{\Gamma(n)} \sum_{l=0}^{\infty} \frac{\Gamma(n+l)}{l!} J_l \,(-\kappa)^l. \tag{4.6}$$

Employing the asymptotic expansion for the exponential-integral function in eq. (4.3) as well as expanding eqs. (4.3) to (4.5) in powers of κ, one can verify the agreement between both representations for $J_n^a(s_0)$.

Let us now turn to a numerical comparison of CI versus FO perturbation theory in the simplified example. To do this, a value for α_s is required and the very recent result $\alpha_s(M_\tau) = 0.34$ [5] will be employed for a_{s_0}. The CI expression of eq. (4.2) and the perturbative expansion of the FO result of eq. (4.6), together with eq. (3.9), then lead to:

$$\begin{aligned}
& \qquad\quad a^0 \qquad\;\; a^1 \qquad\quad\; a^2 \qquad\quad\; a^3 \qquad\quad\; a^4 \qquad\quad\; a^5 \\
R_{\tau,\beta_1}^{D,\mathrm{CI}} &= 3\,[1 + 0.1455 + 0.0305 + 0.0136 + 0.0060 + 0.0031] = 3.5961\,, && (4.7) \\
R_{\tau,\beta_1}^{D,\mathrm{FO}} &= 3\,[1 + 0.1082 + 0.0609 + 0.0254 + 0.0082 + 0.0024] = 3.6154\,. && (4.8)
\end{aligned}$$

From (4.7) and (4.8) one observes that the CI expansion appears to display a better convergence, although the last included term of the FO expansion is somewhat smaller. Altogether, the difference between the two results amounts to $3 \cdot 0.0064$, about 2-3 times the last term in both expansions.

Thus, the agreement between both approaches to performing the renormalisation group improvement is reasonably good. Nevertheless, if we were to assume that all coefficients c_{n1} with $n \geq 6$ vanish identically, the CI result of eq. (4.7) would represent the complete answer. Then the question arises, how the FO result approaches this value at higher orders in perturbation theory. Since with eq. (4.6) an all order expression is at our disposal, the

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1	2	3	4	5	6	7	8
0.108225	0.060933	0.025394	0.008165	0.002419	-0.000607	-0.004193	-0.004669
9	10	11	12	13	14	15	16
-0.000104	0.003728	0.001815	-0.001991	-0.001863	0.000850	0.001407	-0.000275
17	18	19	20	21	22	23	24
-0.000939	0.000032	0.000588	0.000051	-0.000354	-0.000066	0.000209	0.000057
25	26	27	28	29	30	31	32
-0.000122	-0.000043	0.000070	0.000030	-0.000040	-0.000020	0.000023	0.000013

Table 1: Perturbative order n and corresponding terms in the expansion of $R^{D,\mathrm{FO}}_{\tau,\beta_1}/3$.

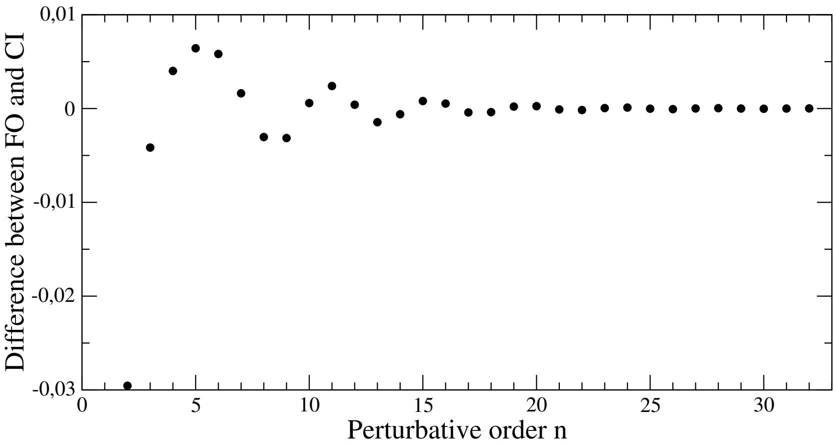

Figure 1: Difference $(R^{D,\mathrm{FO}}_{\tau,\beta_1} - R^{D,\mathrm{CI}}_{\tau,\beta_1})/3$ as a function of the perturbative order n up to which the terms in the FOPT series are summed in the example with only β_1 non-vanishing.

question can be answered by reexpanding (4.6) in powers of a_{s_0}. The numerical result of this exercise is displayed in table 1 up to the 32nd order. The entries are the perturbative order n at the one hand and the corresponding term in the expansion of $R^{D,\mathrm{FO}}_{\tau,\beta_1}/3$ on the other. Therefore, the first five entries agree with the corresponding terms in the square bracket of eq. (4.8).

By inspecting table 1, it is observed that after the first five positive terms, there are four terms with negative sign. After that, the sign changes at about every second term. Generally, the size of the terms decreases with increasing order, but not from one term to the next. A graphical account of these findings is presented in figure 1, which shows the difference $(R^{D,\mathrm{FO}}_{\tau,\beta_1} - R^{D,\mathrm{CI}}_{\tau,\beta_1})/3$ as a function of the perturbative order n up to which the terms in the FOPT series are summed. As can be seen from this figure, the sign changes lead to FO results which oscillate around the CI value. Due to the sign change at about every second term, the oscillation frequency is approximately four perturbative orders. Furthermore, the magnitude of the oscillation is tending towards zero, such that the FO value approaches the CI result at large orders. Calculating the series up to the 80th order, as yet no sign of its asymptotic nature is observed. Incidentally, the maximum of the difference between both approaches is found to appear at the fifth order.

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5. The general case

In order to tackle the general problem with all β-function coefficients β_1 to β_4 being unequal to zero, the general expansion of $a(-s_0x)$ in terms of a_{s_0} is employed, which takes the form:

$$a(-s_0x) = a_{s_0}\left[1+\sum_{i=1}^{\infty}\sum_{j=1}^{i} d_{ij}\, a_{s_0}^i \ln^j(-x)\right]. \tag{5.1}$$

The coefficients d_{ij} can be calculated from the RGE for the QCD coupling eq. (2.4). The coefficients of the highest and the lowest power of the logarithm can easily be given for arbitrary order and read $d_{nn} = (-\beta_1/2)^n$ and $d_{n1} = -\beta_n/2$. The remaining three coefficients up to fourth order are found to be:

$$d_{32} = \frac{5}{8}\beta_1\beta_2\,, \qquad d_{43} = -\frac{13}{24}\beta_1^2\beta_2\,, \qquad d_{42} = \frac{3}{8}(\beta_2^2 + 2\beta_1\beta_3)\,. \tag{5.2}$$

In principle — given the necessary computing resources — the coefficients d_{ij} can be calculated to an arbitrary order, taking into account the known β-function coefficients, and setting the even higher-order ones to zero.

The numerical analysis of the case including all available β-function coefficients proceeds along similar lines as the simplified example in the previous section. For CIPT, now the integrals of eq. (3.10) have to be calculated numerically by plugging a numerical solution of the RGE (2.4) for the running coupling and then performing the contour integration. The FO result can be taken from eq. (3.7). In addition including the fifth-order term, one obtains:

$$\begin{aligned}
&\qquad\quad\; a^0 \qquad a^1 \qquad\quad a^2 \qquad\quad a^3 \qquad\quad a^4 \qquad\quad a^5 \\
R_\tau^{D,\mathrm{CI}} &= 3\,[1 + 0.1479 + 0.0297 + 0.0122 + 0.0047 + 0.0019] = 3.5893\,, \qquad (5.3)\\
R_\tau^{D,\mathrm{FO}} &= 3\,[1 + 0.1082 + 0.0609 + 0.0334 + 0.0144 + 0.0021] = 3.6571\,. \qquad (5.4)
\end{aligned}$$

Again, the CI series displays a better convergence although the last included term in both expansions is almost of the same size. This time, however, the difference between CI and FO perturbation theory looks much more dramatic, as it amounts to $3\cdot 0.0226$, more then ten times the last included term! This apparent ambiguity in the perturbative prediction of the hadronic τ decay rate at the moment represents the dominant theoretical uncertainty in the extraction of the strong coupling α_s from this channel.

To gain further insight into the origin of the problem, like in the previous, simplified example, higher order contributions should be inspected. These can be acquired by employing the expansion (5.1) for the running coupling in the contour integrals of eq. (3.10) which can be expressed in terms of the J_l, and then reexpanding the result in terms of a_{s_0}.[3] The result of this exercise is presented in table 2. Like in the last section, after the first five positive terms, there are four negative ones after which the terms change sign at every third order, and, all in all, decrease in magnitude.

[3] Of course, this is just another way of calculating higher order relations for the c_{nk} coefficients, similar to the ones given in eq. (2.6).

1	2	3	4	5	6	7	8
0.108225	0.060933	0.033422	0.014405	0.002052	-0.006847	-0.012518	-0.012029
9	10	11	12	13	14	15	16
-0.004380	0.005429	0.010012	0.006522	-0.001223	-0.006793	-0.006409	-0.001148
17	18	19	20	21	22	23	24
0.004472	0.004472	0.002420	-0.003022	-0.005577	-0.003107	0.002045	0.005211
25	26	27	28	29	30	31	32
0.003582	-0.001304	-0.004970	-0.004026	0.000713	0.004875	0.004498	-0.000213

Table 2: Perturbative order n and corresponding terms in the expansion of $R_\tau^{D,\mathrm{FO}}/3$.

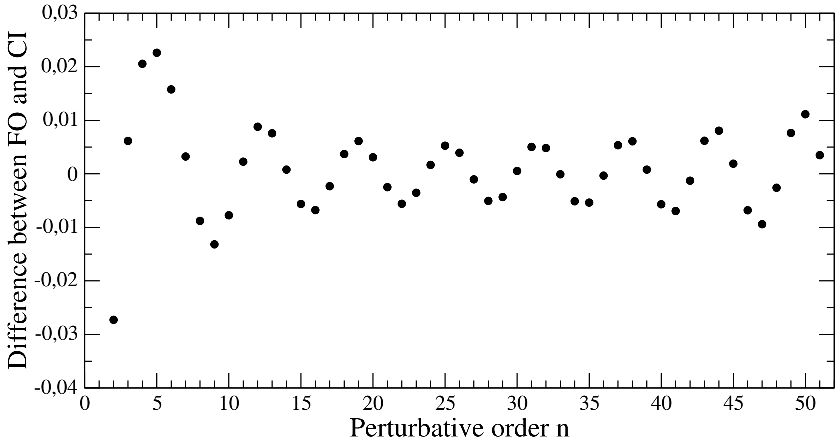

Figure 2: Difference $(R_\tau^{D,\mathrm{FO}} - R_\tau^{D,\mathrm{CI}})/3$ as a function of the perturbative order n up to which the terms in the FOPT series are summed.

Restricting oneself to the five c_{n1} coefficients discussed above as well as the four known β-function coefficients, again the CI value (5.3) represents the full answer. This time, in figure 2 the difference between the FO expansion summed up to a particular order n and the complete CI result is displayed up to the 51st order. Until about the 30th order, the series shows the same behaviour as in the simplified example above. It oscillates around the CI result, while approaching it. However, for the general case the convergence to the CI result is much slower, and roughly after the 30th order, the asymptotic nature of the series sets in and the terms start to increase. The minimal deviation of the FO result at the 30th order is about 0.005, and the oscillation frequency is approximately six perturbative orders. Alas, again the maximum of the deviation occurs at the fifth order.

6. Conclusions

Until today, the extraction of α_s from the hadronic τ decay rate is hampered by an apparent ambiguity between performing the renormalisation group improvement of the perturbative series at a fixed order or in the so-called contour-improved scheme [25, 26]. This ambiguity represents the dominant theoretical uncertainty for α_s as extracted from hadronic τ decays [5].

Further insight into the origin of the problem can be gained by pursuing the fixed-order expansion to larger orders, still staying consistent with the terms that have been resummed in contour-improved perturbation theory. In a simplified example, where only the first coefficient of the β-function, β_1, is kept non-vanishing, this could be done analytically in closed form. For the more general case including the four currently known terms of the β-function, the expansion of the fixed-order result can be calculated to, in principle, arbitrarily high order.

The behaviour of the fixed order series in the simple example as well as in the more general case discussed in sections 4 and 5 have been presented in tables 1 and 2 respectively, and graphically illustrated in figures 1 as well as 2. It is observed that in the full QCD case the fixed-order result oscillates around the contour-improved value with an oscillation frequency of approximately six perturbative orders, converging to it until the 30th order after which the asymptotiv behaviour of the series sets in and the terms again increase. Just including five perturbative terms in the fixed-order case happens to turn out most unfortunate, as for this order the deviation between contour-improved and fixed-order results has a maximum.

All the discussion above demonstrates, that the renormalisation group improvement of perturbative series in finite-energy sum rules, like for the hadronic τ decay rate in particular, should be performed in the contour-improved scheme. This approach also is most natural in the sense that all terms proportional to a particular perturbative coefficient c_{n1} in the correlation function are resummed to yield the contour-improved contribution at order n, whereas in the fixed-order approach terms proportional to the coefficient c_{n1} appear at all orders equal or greater to n. A discussion of renormalisation group improvement of quark-mass corrections in finite-energy sum rules along similar lines will be presented in a forthcoming publication.

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References

[1] E. Braaten, S. Narison and A. Pich, *QCD analysis of the τ hadronic width*, *Nucl. Phys.* **B 373** (1992) 581.

[2] E. Braaten, *The perturbative QCD corrections to the ratio R for tau decay*, *Phys. Rev.* **D 39** (1989) 1458.

[3] E. Braaten, *QCD predictions for the decay of the τ lepton*, *Phys. Rev. Lett.* **60** (1988) 1606.

[4] S. Narison and A. Pich, *QCD formulation of the τ decay and determination of $\Lambda_{\rm MS}$*, *Phys. Lett.* **B 211** (1988) 183.

[5] ALEPH collaboration, S. Schael et al., *Branching ratios and spectral functions of τ decays: final aleph measurements and physics implications*, hep-ex/0506072.

[6] OPAL collaboration, K. Ackerstaff et al., *Measurement of the strong coupling constant α_s and the vector and axial-vector spectral functions in hadronic τ decays*, *Eur. Phys. J.* **C 7** (1999) 571 [hep-ex/9808019].

[7] ALEPH collaboration, R. Barate et al., *Measurement of the spectral functions of axial-vector hadronic τ decays and determination of $\alpha_s(M_\tau^2)$*, *Eur. Phys. J.* **C 4** (1998) 409.

[8] CLEO collaboration, T. Coan et al., *Measurement of α_s from τ decays*, *Phys. Lett.* **B 356** (1995) 580.

[9] ALEPH collaboration, D. Buskulic et al., *Measurement of the strong coupling constant using τ decays*, *Phys. Lett.* **B 307** (1993) 209.

[10] M. Davier, A. Hocker and Z. Zhang, *The physics of hadronic τ decays*, hep-ph/0507078.

[11] OPAL collaboration, G. Abbiendi et al., *Measurement of the strange spectral function in hadronic τ decays*, *Eur. Phys. J.* **C 35** (2004) 437 [hep-ex/0406007].

[12] ALEPH collaboration, R. Barate et al., *Study of τ decays involving kaons, spectral functions and determination of the strange quark mass*, *Eur. Phys. J.* **C 11** (1999) 599 [hep-ex/9903015].

[13] E. Gamiz, M. Jamin, A. Pich, J. Prades and F. Schwab, *V_{us} and m_s from hadronic τ decays*, *Phys. Rev. Lett.* **94** (2005) 011803 [hep-ph/0408044]; *Extraction of m_s and $|V_{us}|$ from hadronic τ decays*, *Nucl. Phys.* **144** *(Proc. Suppl.)* (2005) 59 [hep-ph/0411278].

[14] M. Jamin, *A novel route toV_{us}*, *ECONF* **C0304052** (2003) WG604 [hep-ph/0309147].

[15] E. Gamiz, M. Jamin, A. Pich, J. Prades and F. Schwab, *Determination of m_s and $|V_{us}|$ from hadronic τ decays*, *JHEP* **01** (2003) 060 [hep-ph/0212230].

[16] P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, *Strange quark mass from tau lepton decays with $\mathcal{O}(\alpha_s^3)$ accuracy*, hep-ph/0412350.

[17] S. Chen et al., *Strange quark mass from the invariant mass distribution of cabibbo-suppressed τ decays*, *Eur. Phys. J.* **C 22** (2001) 31 [hep-ph/0105253].

[18] M. Davier, S.M. Chen, A. Hocker, J. Prades and A. Pich, *Strange quark mass from τ decays*, *Nucl. Phys.* **98** *(Proc. Suppl.)* (2001) 319.

[19] J. Kambor and K. Maltman, *The strange quark mass from flavor breaking in hadronic τ decays*, *Phys. Rev.* **D 62** (2000) 093023 [hep-ph/0005156].

[20] J.G. Korner, F. Krajewski and A.A. Pivovarov, *Determination of the strange quark mass from Cabibbo suppressed τ decays with resummed perturbation theory in an effective scheme*, *Eur. Phys. J.* **C 20** (2001) 259 [hep-ph/0003165].

[21] A. Pich and J. Prades, *Strange quark mass determination from Cabibbo-suppressed τ decays*, *JHEP* **10** (1999) 004 [hep-ph/9909244]; *Tau-decay determination of the strange quark mass*, *Nucl. Phys.* **86** *(Proc. Suppl.)* (2000) 236 [hep-ph/9909559].

[22] K.G. Chetyrkin, J.H. Kuhn and A.A. Pivovarov, *Determining the strange quark mass in Cabibbo suppressed τ lepton decays*, *Nucl. Phys.* **B 533** (1998) 473 [hep-ph/9805335].

JHEP09(2005)058

[23] A. Pich and J. Prades, *Perturbative quark mass corrections to the τ hadronic width*, *JHEP* **06** (1998) 013 [hep-ph/9804462]; *Strange quark mass dependence of the τ hadronic width*, *Nucl. Phys.* **74** *(Proc. Suppl.)* (1999) 309 [hep-ph/9811263]; *The strange quark mass dependence of R_τ*, *Nucl. Phys.* **76** *(Proc. Suppl.)* (1999) 341.

[24] N.V. Krasnikov, A.A. Pivovarov and N.N. Tavkhelidze, *The use of finite energy sum rules for the description of the hadronic properties of QCD*, *Z. Physik* **C 19** (1983) 301.

[25] A.A. Pivovarov, *Renormalization group analysis of the τ-lepton decay within QCD*, *Z. Physik* **C 53** (1992) 461 [hep-ph/0302003].

[26] F. Le Diberder and A. Pich, *The perturbative QCD prediction to R_τ revisited*, *Phys. Lett.* **B 286** (1992) 147.

[27] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, *Deep inelastic scattering beyond the leading order in asymptotically free gauge theories*, *Phys. Rev.* **D 18** (1978) 3998.

[28] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, *The Gell-Mann-Low function of QCD in the three loop approximation*, *Phys. Lett.* **B 93** (1980) 429.

[29] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *The four-loop β-function in quantum chromodynamics*, *Phys. Lett.* **B 400** (1997) 379 [hep-ph/9701390].

[30] S.G. Gorishnii, A.L. Kataev and S.A. Larin, *The $\mathcal{O}(\alpha_s^3)$ corrections to $\sigma_{\mathrm{tot}}(e^+e^- \to hadrons)$ and $\Gamma(\tau^- \to \nu_\tau + hadrons)$ in QCD*, *Phys. Lett.* **B 259** (1991) 144.

[31] L.R. Surguladze and M.A. Samuel, *Total hadronic cross-section in e^+e^- annihilation at the four loop level of perturbative QCD*, *Phys. Rev. Lett.* **66** (1991) 560.

[32] P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, *Towards order α_s^4 accuracy in τ decays*, *Phys. Rev.* **D 67** (2003) 074026 [hep-ph/0212299].

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